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LETTER TO THE EDITOR

A q-tensorial approach to q-oscillators in $U_q(su(2))$

D Vrinceanu, M Stroila and A Ludu

Department of Theoretical Physics, University of Bucharest, Bucharest, PO Box MG 5211, Romania

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Abstract. In this letter we construct a q-tensorial structure for q-oscillators. Following an appropriate algebraic formalism we succeeded in constructing classes of tensorial operators and consequently the Casimir element of $U_q(u(2))$, as a 0-rank tensorial operator. The classic Casimir operator of $SU(2)$ is obtained in the limit $q \to 1$.

As was shown in [1] quantum systems admit as internal symmetries more general structures then groups. The quasitriangular Hopf algebras, introduced by Drinfeld [2], represent the appropriate mathematical tools for describing such symmetries.

A particular category of quasitriangular Hopf algebras is supplied by the deformations of the universal enveloping algebra of semisimple Lie algebras [2, 3]. For quantum systems which are approximately described by Lie algebras it is natural to seek as exact symmetry a deformation of these. A good example of this technique is supplied by the IBM method.

Analogous to the classical case where the generators of the Lie algebras are built up with quantum oscillators, it can be proved that the Hopf algebras $U_q(su(n))$ (the deformation of the universal enveloping algebra $U(su(n))$) can be realized with q-oscillators (e.g. $U_q(su(2))$ in [4, 5]). Probably any deformation of the universal enveloping algebra of a semisimple Lie algebra can be obtained in this way.

The creation and annihilation operators of quantum oscillators can be endowed with a spherical tensor structure [8]. In this letter we prove that the creation and annihilation operators of q-oscillators can be endowed with a tensor structure $U_q(su(2))$ (in the sense used in [1]). The philosophical reason for this approach is the fact that any observable of a quantum system must have tensorial properties with respect to the symmetry of the system. One remarks that in all tackled stages, in the limit $q \to 1$, one must obtain classical quantum symmetry.

Quantum symmetry requires:

(i) a Hopf algebra A (see Abe [6]);

(ii) a unitary representation of A on the Hilbert space of states;

(iii) the ground state (the vacuum state) has the transformation law

$$U(g)|0\rangle = \varepsilon(g)|0\rangle \qquad \text{for } g \in A \text{ (unit representation).}$$

(iv) The states belonging to the Hilbert space are created by successive application of some q-tensorial operators on the ground state.

One defines q-tensorial (briefly tensorial) covariant operators as operators defined on the Hilbert space which satisfy the transformation law:

$$U(g)t_m=\sum_p\sum_n D_{nm}(g_p^1)t_nU(g_p^2)$$

where D is a matrix representation of A and

$$\Delta(g)=\sum_p g_p^1\otimes g_p^2.$$

The rank of the q-tensor operator t is the rank of its matrix representation D.

However, it is possible to relax some of these conditions [1].

The precise Hopf algebra in which we are strongly interested is $U_q(su(2))$. This is an associative algebra with generators J_+, J_-, $q^{\pm H/2}$, 1, defined by the relations:

$$[J_+,J_-]=[H] \qquad q^{H/2}q^{-H/2}=q^{-H/2}q^{H/2}=1$$
$$q^{H/2}J_\pm q^{-H/2}=q^{\pm}J_\pm.$$

It is easy to see that $U_q(su(2))$ is a Hopf algebra with:

(i) the coproduct

$$\Delta(J_\pm)=J_\pm\otimes q^{H/2}+q^{-H/2}\otimes J_\pm$$
$$\Delta(q^{\pm H/2})=q^{\pm H/2}\otimes q^{\pm H/2}$$

(ii) the co-unit

$$\varepsilon(J_\pm)=0 \qquad \varepsilon(q^{\pm H/2})=1$$

(iii) the antipode

$$S(J_\pm)=-q^{\pm}J_\pm \qquad S(q^{\pm H/2})=q^{\mp H/2}.$$

Its finite-dimensional irreducible representations are given by integer or by half-integer number l. If the representation D^l corresponds to the integer l then the carrier space V_l of D^l has the orthonormal basis e_m, $m=-l,-l+1,\ldots,l$, such that:

$$U(J_\pm)e_m=([l\mp m][l\pm m+1])^{1/2}e_{m\pm 1} \qquad He_m=me_m$$

with $[n]$ given by:

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}}.$$

The representations D^l of $U_q(su(2))$ are deformation of the corresponding representations of the classical Lie algebra $su(2)$. Finite-dimensional representations of $U_q(su(2))$ are completely reducible.

As in the classical case we have the expansion:

$$|ln\rangle=\sum_{m+m'=n}\langle jmkm'|ln\rangle|jm\rangle\otimes|km'\rangle$$

which defines q-Clebsch–Gordan coefficients of the tensor product $D^j\otimes D^k$, see [7].

We can also prove, as in the classical case:

$$\sum_p\sum_{m+m'=n}\langle jmkm'|ln\rangle D^j_{rm}(g_p^1)D^k_{sm'}(g_p^2)=\sum_0 D^l_{0n}(g)\langle jrks|l0\rangle.$$

Proposition 1. The tensorial product of tensorial operators is a tensorial operator.

Proof. Let t_m be a covariant tensor operator of rank j and $t'_{m'}$ a covariant tensor operator of rank k. Then the tensorial product of these two tensorial operators is:

$$T_n=(t\otimes t')^l_n=\sum_{m+m'=n}\langle jmkm'|ln\rangle t_m t'_{m'}$$

where the rank l runs in the same range as in the classical case.

We have:

$$U(g)T_n=\sum_{m+m'=n}\langle jmkm'|ln\rangle\sum_{p,q}\sum_{r,s}D^j_{rm}(g^1_p)D^k_{sm'}(g^{21}_{pq})t_r t'_s U(g^{22}_{pq}).$$

If we use the co-associativity property, i.e.

$$(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta$$

we obtain

$$\begin{aligned}U(g)T_n&=\sum_{m+m=n}\langle jmkm'|ln\rangle\sum_{r,s}\sum_{p,q}D^j_{rm}(g^{11}_{pq})D^k_{sm'}(g^{12}_{pq})t_r t'_s U(g^2_p)\\&=\sum_{r,s}\sum_{p}D^l_{0n}(g^1_p)\langle jrks|l0\rangle t_r t'_s U(g^2_p)\\&=\sum_{p}D^l_{0n}(g^1_p)T_0U(g^2_p).\end{aligned}$$

Thus T_n is a tensorial operator of rank l. □

Proposition 2. Any scalar tensorial operator is an invariant operator.

Proof. If C is a scalar operator, that is C is a 0-rank tensor operator, then it satisfies the transformation law given by:

$$U(g)C=\sum_p D^0(g^1_p)CU(g^2_p).$$

The 0-rank representation is the representation which transforms ground state, that is ε. Then

$$\sum_p D^0(g^1_p)CU(g^2_p)=\sum_p\varepsilon(g^1_p)CU(g^2_p)=CU\left(\sum_p\varepsilon(g^1_p)g^2_p\right).$$

Next, using the co-unit property, i.e. $(\varepsilon\otimes\mathrm{id})\Delta=\mathrm{id}$, we obtain:

$$U(g)C=CU(g).$$ □

Proposition 3. If T_n is a l-rank tensorial operator then the states $T_n|0\rangle$ are basic states of the rank l representation.

The proof is obvious.

Let us consider an operator b and its adjoint b^+, which acts on the Hilbert space, with a base $|n\rangle$, $n=0, 1, 2, \ldots$, like:

$$b|0\rangle=0 \qquad b|n\rangle=([n])^{1/2}|n-1\rangle \qquad \text{and} \qquad b^+|n\rangle=([n+1])^{1/2}|n+1\rangle.$$

The operator N is defined by $N|n\rangle=n|n\rangle$.

With these definitions we have the following relations:

$$bb^+ = [N+1] \qquad b^+b = [N] \qquad [N, b^+] = b^+ \qquad [N, b] = -b$$
$$q^{rN} b q^{-rN} = q^{-r} b \qquad q^{rN} b^+ q^{-rN} = q^r b^+$$

for any real r, and the q-bosonic commutation relation

$$bb^+ - q b^+ b = q^{-N} \qquad \text{or} \qquad bb^+ - q^{-1} b^+ b = q^N.$$

Following [4] we construct the generators of $U_q(su(2))$ with two independent q-oscillators:

$$J_+ = b_1^+ b_2 \qquad J_- = b_2^+ b_1 \qquad H = N_1 - N_2.$$

Consequently Hopf algebra $U_q(su(n))$ can be obtained with n independent q-oscillators.

We now consider tensorial operators constructed with q-oscillators.

We know that $2j+1$ independent quantum oscillator operators can be endowed with spherical tensor structure in the following procedure [8, 9]:

$$\tilde{b}^+_{jm} = b^+_{jm} \qquad \text{and} \qquad \tilde{b}_{jm} = (-1)^{j-m} b_{j,-m} \qquad (m = -j, -j+1, \ldots, j).$$

In the following we show how the pair of q-oscillators which generate the Hopf algebra $U_q(su(2))$ can be endowed, also, with q-tensor structure.

Thus, we consider two bosonic states (with the quantum number of the angular momentum $\frac{1}{2}$) characterized by the operators b and b^+ with index 1 when we refer to $\frac{1}{2}$ states, and with index 2 when we refer to $-\frac{1}{2}$ states. A direct verification shows us that not only must b be altered but b^+ also.

Finally we found the next tensor operators:

$$\tilde{b}_1^+ = q^{-N_2/2} b_1^+ \qquad \tilde{b}_2^+ = q^{N_1/2} b_2^+$$
$$\tilde{b}_1 = q^{(N_1+1)/2} b_2 \qquad \tilde{b}_2 = -q^{-(N^2+1)/2} b_1. \qquad (*)$$

Observations:

(1) The most general form for these operators is:

$$\tilde{b}_1^+ = q^{-N_2/2} b_1^+ \qquad \tilde{b}_2^+ = q^{N_1/2} b_2^+$$
$$\tilde{b}_1 = q^{\alpha} q^{N_1/2} b_2 \qquad \tilde{b}_2 = -q^{\beta} q^{-N_2/2} b_1$$

with $\alpha - \beta = 1$. We choose the $(*)$ form ($\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$) for symmetry reasons.

(2) When $q \to 1$ one turns to the classical situation. Using proposition 1 we construct the tensor operators of rank 0 and 1, bilinear in the elementary operators b. The scalar operator $\mathcal{N}$ is defined:

$$\mathcal{N} = (\tilde{b}^+ \otimes \tilde{b})^0 = \sum \langle \tfrac{1}{2} m \tfrac{1}{2} m' | 00 \rangle \tilde{b}^+_m \tilde{b}_{m'}.$$

We obtain

$$\mathcal{N} = -[N]/([2])^{1/2} \qquad \text{where } N = N_1 + N_2.$$

So, using proposition 2, it results that $\mathcal{N}$ is an invariant operator. The 1-rank tensor operators T are given by:

$$T_n = (\tilde{b}^+ \otimes \tilde{b})^1_n = \sum \langle \tfrac{1}{2} m \tfrac{1}{2} m' | 1 n \rangle \tilde{b}^+_m \tilde{b}_{m'}.$$

Then we have:

$$T_+ = T_1 = q^{H/2} J_+$$
$$T_- = T_{-1} = -q^{H/2} J_-$$
$$T_0 = ([2])^{-1/2}(q^{N_1+1}[N_2] - q^{-(N_2+1)}[N_1]).$$

These operators allow us to build up the invariant operator $\mathscr{C}$ (quadratic in elementary operators b—two body interactions).

$$\mathscr{C} = (T \otimes T)^0 = \sum \langle 1m1m'|00\rangle T_m T_{m'}.$$

Then $\mathscr{C}$ shows:

$$\mathscr{C} = ([3])^{-1/2}(qT_+T_- + q^{-1}T_-T_+ - T_0^2).$$

To find eigenvalues of this new q-Casimir, we put $\mathscr{C}$ in the form:

$$\mathscr{C} = -([3])^{-\frac{1}{2}}\left(2q^H J_- J_+ + q^H[H] + \frac{1}{[2]}\left(\frac{q^{N+1}+q^{-N-1}}{q-q^{-1}} - q^H \frac{q+q^{-1}}{q-q^{-1}}\right)^2\right).$$

If one acts with $\mathscr{C}$ on the maximal weight state $|jj\rangle$ one obtains eigenvalues for fixed N:

$$c(N,j) = -([3])^{-\frac{1}{2}}\left(q^{2j}[2j] + \frac{1}{[2]}\left(\frac{q^{N+1}+q^{-N-1}}{q-q^{-1}} - q^{2j}\frac{q+q^{-1}}{q-q^{-1}}\right)^2\right).$$

We notice that in the above expression the quantum numbers N and j are mixed. In the limit $q \to 1$ this mixing falls and the eigenvalues are the well known $j(j+1)$ (up to a factor).

In conclusion we presented a q-tensorial structure of q-oscillators. This was suggested by the IBM method [10]. The q-tensor structure of q-oscillators leads to a realization of q-tensorial operators. Finally we pointed out some new directions. Further on one can try to endow directly the operators b^+_{lm}, b_{lm} (for $l > \frac{1}{2}$) with a tensorial structure. This would help to construct a whole deformation of the chains of subalgebras used in the IBM method [11]. On the other hand we suggest working with a new tensor creation operator:

$$\tilde{p}^+_{lm} = (\tilde{b}^+ \otimes \ldots \otimes \tilde{b}^+)^l_m.$$

Thus there appears the possibility of building up the subalgebra chains and their invariants in terms of such operators.

With regard to the $(*)$ relations, for $|q| = 1$, the deformation factors can be interpreted as phase factors. At the same time, in this condition one defines the contragradient representation with the antipode S and contravariant operators. From these observations we suppose that $|q| = 1$ is the adequate deformation.

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